Eur. Phys. J. C 6, 549–553 (1999)
DOI 10.1007/s100529800933

THE EUROPEAN PHYSICAL JOURNAL C

Summing up the perturbation series in the Schwinger Model

Tomasz Radożycki[a]

Physics Department, Warsaw University, ul. Hoża 69, 00-681 Warsaw, Poland

Received: 2 March 1998 / Revised version: 12 May 1998 / Published online: 8 September 1998

Abstract. Perturbation series for the electron propagator in the Schwinger Model is summed up in a direct way by adding contributions coming from individual Feynman diagrams. The calculation, performed entirely in momentum space, shows the complete agreement between nonperturbative and perturbative approaches.

A long-standing question in quantum field theory is the connection between perturbation series and exact, nonperturbative results. It dates back to the Dyson's paper [1] in which the author, considering stability of a system conditions, suggested that physical quantities and Green's functions should be nonanalytic in the coupling constant g around $g = 0$. This in turn should result in the divergence, usually factorial type, of the perturbation series. This conjecture was supported by simple models [2] among which the most widely considered was the anharmonic oscillator and its field-theoretical counterpart — the ϕ^4 theory [3–11] as well as by other, more realistic, field theories as QED for instance [12,13] (see [14] for further references). In these cases the required estimations for the nonperturbative results were often obtained with the use of the generalized (Padé, Borel) summation methods (for a review of this approach see [14–16]). There have also been found counterexamples, regarding the Dyson's observation, in which the perturbation series is not divergent in spite of instability (although it may be convergent to an incorrect result) [17–19].

In QED the nonanalyticity in the coupling constant often manifests itself through the presence of a logarithmic function of the fine structure constant α in the calculated quantities [20–22] and in consequence means the divergence of coefficients in the Taylor expansion in α (in other words divergence of Feynman diagrams) resulting in necessity of infinite renormalisation. One can say that this means the incorrectness of the perturbation expansion [23–30].

Although the summability of the perturbation series still remains an opened question, perturbation theory constitutes, however, the main tool in practical calculations giving, especially in Quantum Electrodynamics, excellent results. It seems, therefore, valuable to sum up directly the perturbation series, by adding contributions of the individual Feynman graphs, in a model theory in which the nonperturbative result is well known. Also the very technique for performing this summation is interesting in itself. In this work we will concentrate on the 1+1 dimensional massless QED known as the Schwinger Model [31]. In other two-dimensional models like ϕ^4, Yukawa theory or QCD various and model dependent behaviours of the perturbation series have been established [32–34]

The focus will be put here on the electron propagator for which the explicit nonperturbative formula in coordinate space was found [31] (up to the final p-integration)

$$S(x) = \mathcal{S}_0(x) \exp\left[-ie^2\beta(x)\right] \ , \tag{1}$$

$\mathcal{S}_0$ being the free propagator. Function β is defined by

$$\beta(x) = \begin{cases} \frac{i}{2e^2}\left[-\frac{i\pi}{2} + \gamma_E + \ln\sqrt{e^2x^2/4\pi} + \frac{i\pi}{2}H_0^{(1)}(\sqrt{e^2x^2/\pi})\right] \\ \qquad\qquad x \quad \text{timelike} \\ \frac{i}{2e^2}\left[\gamma_E + \ln\sqrt{-e^2x^2/4\pi} + K_0(\sqrt{-e^2x^2/\pi})\right] \\ \qquad\qquad x \quad \text{spacelike} \end{cases} . \tag{2}$$

Symbol γ_E denotes here the Euler constant and functions $H_0^{(1)}$ and K_0 are Hankel function of the first kind, and Basset function respectively [35].

The propagator S was also considered perturbatively [36,37] but the calculations were led in coordinate space. In that case, thanks to the well known factorization, the perturbation series is either turned into a differential equation or can be shown to be the expansion of the exponent function. In real four-dimensional QFT, however, one most often has to do with Feynman diagrams in momentum space, where S-matrix elements have their natural form and the analytical properties of the Green's functions (poles, cuts) are closely related to the physical quantities. We therefore find it more instructive to sum up the perturbation series entirely in momentum space. Up to our knowledge no such direct summation has, in this model, been performed.

One can expect that in this case the perturbation series should be convergent and give, as the sum, the correct result since:

[a] E-mail: torado@fuw.edu.pl

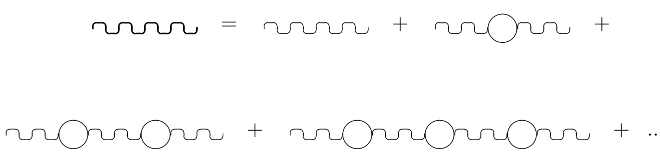

Fig. 1. Diagrams contributing to the photon propagator

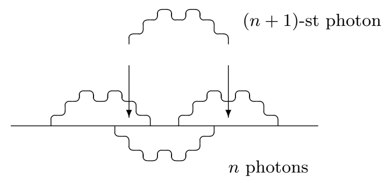

Fig. 2. The attachment of the $(n+1)$-st photon to $S^{(n)}(p)$

1. no infinite renormalisation has to be performed in the model
2. if one reverses the sign of e^2, as suggested by Dyson, no collapse should arise since in two dimensions the potential between equal sign charges would be bounded from below
3. the appearance of a logarithm in (2) is only apparent as the Hankel and Basset functions for small arguments — which means small values of the coupling constant (or small distances which is equivalent here as the scale in the theory is imposed by e) — behave like

$$H_0^{(1)}(z) \approx \frac{2i}{\pi}(\ln x/2 + \gamma_E) + 1 + \text{analytic terms} \ , \quad (3)$$

and similarly for the K_0 function

$$K_0(z) \approx -\ln x/2 - \gamma_E + \text{analytic terms} \ , \quad (4)$$

and nonanalytic functions cancel each other. The full propagator turns out to be the free one in this limit (which corresponds also to the UV limit).

The Schwinger Model may be characterized by the Lagrangian density

$$\mathcal{L}(x) = \overline{\Psi}(x)\left[i\gamma^\mu\partial_\mu - eA^\mu(x)\gamma_\mu\right]\Psi(x) \\ -\frac{1}{4}F^{\mu\nu}(x)F_{\mu\nu}(x) - \frac{\lambda}{2}\left[\partial_\mu A^\mu(x)\right]^2 \ . \quad (5)$$

The parameter λ is here a gauge fixing one, and later will be set to infinity which corresponds to the choice of the Landau gauge.

In order to sum the perturbation series for the electron propagator one first has to perform a presummation of vacuum polarization diagrams. It is well known that this presummation is trivial since in this simple model fermion loops with more than two vertices do not contribute and only diagrams of Fig. 1 should be taken into account. It may be easily checked by an explicit calculation that for a single loop one gets

$$\Pi^{\mu\nu}(k) = ie^2\int\frac{d^2p}{(2\pi)^2}\mathrm{Tr}\left(\gamma^\mu\frac{1}{\not p + i\varepsilon}\gamma^\nu\frac{1}{\not p + \not k + i\varepsilon}\right) \\ = \frac{e^2}{\pi}\left(g^{\mu\nu} - \frac{k^\mu k^\nu}{k^2}\right) \ , \quad (6)$$

so that the whole series of Fig. 1 may be easily summed up to give the massive propagator

$$D^{\mu\nu}(k) = \left(-g^{\mu\nu} + \frac{k^\mu k^\nu}{k^2}\right)\frac{1}{\mu^2 - k^2} - \frac{1}{\lambda}\frac{k^\mu k^\nu}{(k^2)^2} \ , \quad (7)$$

with $\mu^2 = e^2/\pi$. This is the famous Schwinger boson.

Now we have to consider electron self-energy insertions assuming already that we have to do with massive photons. This summation is not trivial and we will perform it in detail. Let us represent the full propagator S, in momentum space, as the sum

$$S(p) = \sum_{n=0}^{\infty} S^{(n)}(p) \ , \quad (8)$$

where $S^{(0)}$ is of course the same as Fourier transformed $\mathcal{S}_0(x)$ of (1), and the summation runs over the number of photons attached to the electron line. To find the recurrent relation between $S^{(n)}$'s we take the n-th term of the sum (8) and attach to it the $(n+1)$-st photon. This situation is schematically represented on Fig. 2. When the photon is attached the additional propagator $D^{\mu\nu}(k)$ appears in the internal line. It may easily be observed that this part of $D^{\mu\nu}$ that bears metric tensor $g^{\mu\nu}$ does not contribute since the corresponding expression has the structure

$$\gamma^\mu\gamma^{\alpha_1}\gamma^{\alpha_2}\cdot\ldots\cdot\gamma^{\alpha_{2k+1}}\gamma_\mu \quad (9)$$

and an odd number of gamma matrices may, in two dimensions, always be reduced to only one for which one can check that $\gamma^\mu\gamma^\alpha\gamma_\mu = 0$. For the gamma matrices we use in this work the following convention

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \ , \quad \gamma^1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \ , \\ \gamma^5 = \gamma^0\gamma^1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \ ,$$

and for the metric tensor: $g^{00} = -g^{11} = 1$.

Thanks to this observation we may now consider only that part of $D^{\mu\nu}(k)$ which is proportional to $k^\mu k^\nu$

$$-\frac{k^\mu k^\nu}{(k^2 - \kappa^2)(k^2 - \mu^2)} \ , \quad (10)$$

where we have put $\lambda \to \infty$, and introduced fictious mass κ^2 in denominator to avoid infinities at intermediate steps when we separate the k-integral into pieces. Let us now imagine that we first attach to the object $S^{(n)}(p)$ only one leg of the external photon of the (incoming) momentum k. This means that we consider the vertex in the n-th order: $[S(p+k)\Gamma^\mu(k,p)S(p)]^{(n)}$. But our (simplified) propagator (10) provides also k^μ in this vertex. From the very construction of the theory and its gauge symmetry it follows that in each order the Ward identity is separately satisfied which may also be checked by a direct computation

$$k_\mu \left[S(p+k)\Gamma^\mu(k,p)S(p)\right]^{(n)} = S^{(n)}(p) - S^{(n)}(p+k) \; . \tag{11}$$

If we now attach to the above object the second photon leg (now of momentum $-k$) we obtain

$$[S(p-k)\Gamma^\nu(-k,p)S(p) - S(p)\Gamma^\nu(-k,p+k)S(p+k)]^{(n)} \; . \tag{12}$$

The second leg, according to (10), also bears k_ν so we can use again the Ward identity getting

$$\begin{aligned} &k_\nu \left[S(p-k)\Gamma^\nu(-k,p)S(p) - S(p)\Gamma^\nu(-k,p+k)S(p+k)\right]^{(n)} \\ &\quad = S^{(n)}(p-k) - S^{(n)}(p) + S^{(n)}(p+k) - S^{(n)}(p) \; . \end{aligned} \tag{13}$$

Now we are in a position to state our recurrence equation between $S^{(n)}$'s

$$\begin{aligned} S^{(n+1)}(p) = &-(-ie)^2 \frac{i}{2(n+1)} \\ &\times \int \frac{d^2k}{(2\pi)^2} \frac{1}{(k^2-\mu^2+i\varepsilon)(k^2-\kappa^2+i\varepsilon)} \\ &\times \left[2S^{(n)}(p-k) - 2S^{(n)}(p)\right] \; , \end{aligned} \tag{14}$$

where in one of the terms in (14) we have changed $k \to -k$ under the integral. The combinatorical factor $\frac{1}{2(n+1)}$ comes from the fact that our construction counts each diagram $2(n+1)$ times ($n+1$ possibilities of the choice which photon we treat as the $(n+1)$-st one and two possibilities of interchanging the attached legs). Finally we can write this equation in the form [38]

$$\begin{aligned} S^{(n+1)}(p) = \frac{ie^2}{n+1} \Bigg[&-\mathcal{I}(\mu^2,\kappa^2) S^{(n)}(p) + \int \frac{d^2k}{(2\pi)^2} \\ &\times \frac{1}{(k^2-\mu^2+i\varepsilon)(k^2-\kappa^2+i\varepsilon)} S^{(n)}(p-k) \Bigg] \; , \end{aligned} \tag{15}$$

where for convenience symbol $\mathcal{I}$ has been introduced to denote

$$\begin{aligned} \mathcal{I}(\mu^2,\kappa^2) &\equiv \int \frac{d^2k}{(2\pi)^2} \frac{1}{(k^2-\mu^2+i\varepsilon)(k^2-\kappa^2+i\varepsilon)} \\ &= \frac{i}{4\pi} \ln \frac{\mu^2/\kappa^2}{\mu^2-\kappa^2} \; . \end{aligned} \tag{16}$$

One could observe in this point that the passing to the coordinate space would simplify further calculations since the convolution integral on the right hand side of (15) would change into product and $S^{(n)}$ would turn out to be the n-th term of the expansion of the exponent function which allows for an easy summation. As motivated in the introduction we prefer to proceed in momentum space which may be an earnest (naturally strongly simplified) of what one can have to do with in perturbative QED$_4$.

Repeating the recurrence we are able to write the general formula for the n-th term

$$\begin{aligned} S^{(n)}(p) = &\frac{(ie^2)^n}{n!} \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} [-\mathcal{I}(\mu^2,\kappa^2)]^{(n-k)} \\ &\times \int \frac{d^2k_1 d^2k_2 \cdot \ldots \cdot d^2k_k}{(2\pi)^{2k}} \\ &\cdot \frac{1}{(k_1^2-\mu^2+i\varepsilon)(k_1^2-\kappa^2+i\varepsilon)} \\ &\cdot \frac{1}{(k_2^2-\mu^2+i\varepsilon)(k_2^2-\kappa^2+i\varepsilon)} \cdot \ldots \\ &\cdot \frac{1}{(k_k^2-\mu^2+i\varepsilon)(k_k^2-\kappa^2+i\varepsilon)} \cdot \\ &S^{(0)}(p-k_1-k_2-\ldots-k_k) \; . \end{aligned} \tag{17}$$

Considering the summation in (8) together with that of formula (17) we see that the double sum has to be performed. Using obvious symbolic notation we can simplify it in the following way

$$\begin{aligned} &\sum_{n=0}^{\infty} \frac{x^n}{n!} \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} a_k \\ &= \sum_{k=0}^{\infty} \frac{a_k}{k!} \sum_{n=k}^{\infty} \frac{x^n}{(n-k)!} = \sum_{k=0}^{\infty} \frac{a_k x^k}{k!} \sum_{n=k}^{\infty} \frac{x^{n-k}}{(n-k)!} \\ &= \sum_{k=0}^{\infty} \frac{a_k x^k}{k!} \sum_{n=0}^{\infty} \frac{x^n}{n!} = \mathrm{e}^x \sum_{k=0}^{\infty} \frac{a_k x^k}{k!} \; . \end{aligned} \tag{18}$$

Applying this to our formula for $S(p)$ we get

$$\begin{aligned} S(p) = &\exp\left[-ie^2\mathcal{I}(\mu^2,\kappa^2)\right] \sum_{n=0}^{\infty} \frac{(ie^2)^n}{n!} \\ &\times \int \frac{d^2k_1 d^2k_2 \cdot \ldots \cdot d^2k_n}{(2\pi)^{2n}} \\ &\cdot \frac{1}{(k_1^2-\mu^2+i\varepsilon)(k_1^2-\kappa^2+i\varepsilon)} \\ &\cdot \frac{1}{(k_2^2-\mu^2+i\varepsilon)(k_2^2-\kappa^2+i\varepsilon)} \cdot \ldots \\ &\cdot \frac{1}{(k_n^2-\mu^2+i\varepsilon)(k_n^2-\kappa^2+i\varepsilon)} \\ &\cdot S^{(0)}(p-k_1-k_2-\ldots-k_n) \; . \end{aligned} \tag{19}$$

We now have to make use of the fact that $S^{(0)}(p)$ is a free massless propagator: $S^{(0)}(p) = \gamma^\mu p_\mu/p^2$, pass to the Euclidean space, and replace denominators $1/D^2$ with $\int_0^\infty dt$ $\exp[-tD^2]$. If we additionally substitute for $p^\mu - k_1^\mu - \ldots - k_n^\mu$ the appropriate derivative over p_μ we can write

$$\begin{aligned} S(p)_E = &\frac{1}{2} \exp\left[-ie^2\mathcal{I}(\mu^2,\kappa^2)\right] \left(\gamma_\mu \frac{\partial}{\partial p_\mu}\right)_E \\ &\times \sum_{n=0}^{\infty} \frac{(-e^2)^n}{n!} \frac{1}{(\mu^2-\kappa^2)^n} \int_0^\infty \frac{d\tau}{\tau} \int_0^\infty dt_1 dt_2 \cdot \ldots \cdot dt_n \end{aligned}$$

$$\cdot \int \frac{d_E^2k_1 d_E^2k_2 \cdot ... \cdot d_E^2k_n}{(2\pi)^{2n}} \sum_{i=0}^{n} \frac{n!}{i!(n-i)!}(-1)^i$$
$$\times \exp\big[-t_1(k_1^2+\mu^2) - t_2(k_2^2+\mu^2) - ...$$
$$- t_i(k_i^2+\mu^2) - t_{i+1}(k_{i+1}^2+\kappa^2) - ...$$
$$- t_n(k_n^2+\kappa^2) - \tau(p-k_1-k_2-...-k_n)^2\big] . \quad (20)$$

In this formula the coefficient $1/(\mu^2-\kappa^2)^n$ arises from expanding the products of denominators $1/[(k_i^2+\mu^2)(k_i^2+\kappa^2)]$ into sums. Now we calculate the multiple integral

$$\int \frac{d_E^2k_1 d_E^2k_2 \cdot ... \cdot d_E^2k_n}{(2\pi)^{2n}} \exp\big[-t_1k_1^2 - t_2k_2^2 - ... - t_nk_n^2$$
$$-\tau(p-k_1-k_2-...-k_n)^2\big]$$
$$= \frac{1}{(4\pi)^n}\frac{1}{(1/\tau+1/t_1+...+1/t_n)\,\tau t_1\cdot ... \cdot t_n}$$
$$\times \exp\left[-p^2/\left(1/\tau+1/t_1+...+1/t_n\right)\right] . \quad (21)$$

After having taken in (20) the derivative over p_μ the integral over τ may be easily performed if we observe that

$$\frac{1}{(1/\tau+x)^2\,\tau^2}\exp\left[-\frac{p^2}{1/\tau+x}\right] = -\frac{1}{p^2}\frac{d}{d\tau}\exp\left[-\frac{p^2}{1/\tau+x}\right]$$

and one limit contributes $\frac{1}{p^2}$ and the other $-\frac{1}{p^2}\mathrm{e}^{-p^2/x}$. In that way we obtain for $S(p)_E$

$$S(p)_E = \exp\left[-ie^2\mathcal{I}(\mu^2,\kappa^2)\right]\frac{(\gamma_\mu p^\mu)_E}{p^2}$$
$$\times \sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{-e^2}{4\pi(\mu^2-\kappa^2)}\right)^n \int_0^\infty dt_1 dt_2\cdot ... \cdot dt_n \cdot$$
$$\exp\left[-\kappa^2(t_1+t_2+...+t_n)\right]$$
$$\times\left(\exp\left[-\frac{p^2}{1/t_1+1/t_2+...+1/t_n}\right]-1\right)$$
$$\times\sum_{i=0}^{n}\frac{n!}{i!(n-i)!}(-1)^i\,\cdot$$
$$\cdot\exp\left[-(\mu^2-\kappa^2)(t_1+t_2+...+t_i)\right] . \quad (22)$$

Now let us consider the expression under the second sum

$$\sum_{i=0}^{n}\frac{n!}{i!(n-i)!}(-1)^i f(t_1)f(t_2)\cdot ... \cdot f(t_i) .$$

Since it will be integrated in (22) over all t_i's with a symmetric function of its arguments one can obviously replace it with

$$[1-f(t_1)]\cdot[1-f(t_2)]\cdot ... \cdot[1-f(t_n)]$$

and that, in turn, leads to

$$S(p)_E = \exp\left[-ie^2\mathcal{I}(\mu^2,\kappa^2)\right]\frac{(\gamma_\mu p^\mu)_E}{p^2}$$
$$\times\sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{-e^2}{4\pi(\mu^2-\kappa^2)}\right)^n\int_0^\infty dt_1dt_2\cdot ... \cdot dt_n$$
$$\times\frac{\mathrm{e}^{-\kappa^2t_1}-\mathrm{e}^{-\mu^2t_1}}{t_1}\cdot\frac{\mathrm{e}^{-\kappa^2t_2}-\mathrm{e}^{-\mu^2t_2}}{t_2}$$
$$\cdot ... \cdot\frac{\mathrm{e}^{-\kappa^2t_n}-\mathrm{e}^{-\mu^2t_n}}{t_n}\cdot$$
$$\cdot\left(\exp\left[-\frac{p^2}{1/t_1+1/t_2+...+1/t_n}\right]-1\right) . \quad (23)$$

Making now use of the identity which is valid for $a>0$ [39]

$$1-\mathrm{e}^{-1/4a} = \int_0^\infty dx J_1(x)\mathrm{e}^{-ax^2} ,$$

where J_1 is the Bessel function, together with the substitution:

$$\frac{1}{4a} = \frac{p^2}{\left(\frac{1}{t_1}+\frac{1}{t_2}+...+\frac{1}{t_1}\right)} ,$$

we note that we are now in a position to perform all t_i integrations according to

$$\int_0^\infty dt\frac{\mathrm{e}^{-\kappa^2t}-\mathrm{e}^{-\mu^2t}}{t}\mathrm{e}^{-x^2/4p^2t}$$
$$= 2\left[K_0\left(\frac{\kappa x}{\sqrt{p^2}}\right)-K_0\left(\frac{\mu x}{\sqrt{p^2}}\right)\right] .$$

Observing that in (23) we have in fact the expansion of the exponent function we can write down the following formula

$$S(p)_E \quad (24)$$
$$= -\exp\left[-ie^2\mathcal{I}(\mu^2,\kappa^2)\right]\frac{(\gamma_\mu p^\mu)_E}{p^2}\int_0^\infty dx J_1(x)$$
$$\times\exp\left\{-\frac{e^2}{2\pi(\mu^2-\kappa^2)}\left[K_0\left(\frac{\kappa x}{\sqrt{p^2}}\right)-K_0\left(\frac{\mu x}{\sqrt{p^2}}\right)\right]\right\} .$$

The quantity κ was introduced to the calculations only temporarily in order to regularize certain integrals on intermediate steps. Now, in the formula (24), where all pieces are collected together, we may get rid of it, setting $\kappa\to 0$, if we make use of the expansion of Basset function for small arguments: $K_0(x)\approx -\ln(x/2)-\gamma_E$. Recalling that $\mu^2 = e^2/\pi$ we finally get

$$S(p)_E = -\frac{(\not{p})_E}{(p^2)^{5/4}}\mathrm{e}^{\gamma_E/2}\left(\frac{e}{2\sqrt{\pi}}\right)^{1/2}$$
$$\times\int_0^\infty dx x^{1/2}J_1(x)\exp\left[\frac{1}{2}K_0\left(ex/\sqrt{\pi p^2}\right)\right] . \quad (25)$$

If one takes into account the asymptotic approximation of the function K_0 one can easily obtain the known [40] infrared behaviour of the electron propagator (in Minkowski space): $S(p)\approx\frac{e^{1/2}}{2^{5/2}\pi^{5/4}}\exp\left(\frac{\gamma_E}{2}\right)\left[\Gamma\left(\frac{1}{4}\right)\right]^2\frac{\not{p}}{(-p^2)^{5/4}}$.

Then we already have the Euclidean p representation of S which, in fact, terminates the caculations, but in order to compare the result with (1) and (2) we also need the coordinate space representation. The lacking Fourier transform may, however, be performed in a straightforward way described below. After rescaling $x \to x \cdot p$, where $p = \sqrt{p^2}$, replacing p^μ/p with $\partial/\partial p_\mu$, which acts on the function of p only, and noticing that $J_1(x) = -dJ_0(x)/dx$ one gets

$$S(p)_E = \left(\gamma_\mu \frac{\partial}{\partial p_\mu}\right)_E \mathrm{e}^{\gamma_E/2} \left(\frac{e}{2\sqrt{\pi}}\right)^{1/2} \times \int_0^\infty dx x^{-1/2} J_0(xp) \exp\left[\frac{1}{2} K_0\left(ex/\sqrt{\pi}\right)\right] . \quad (26)$$

The following representation for the Bessel function $J_0(x)$

$$J_0(xp) = \frac{1}{2\pi} \int_0^{2\pi} \mathrm{e}^{ipx \sin(\phi - \alpha)} d\phi$$

can now be used, where we choose the angle α such that: $\cos\alpha = p_4/p$, and $\sin\alpha = p_1/p$. After this substitution our formula (26) contains two integrations: over x and ϕ and they may be replaced with the integration over Euclidean two-space if we identify

$$x_4 = x \sin\phi , \qquad x_1 = x \cos\phi .$$

Taking into account that the appropriate Jacobian equals $1/x$ and passing to Minkowski space-time we can finally write down

$$S(p) = -\frac{1}{2\pi} e^{\gamma_E/2} \int d^2x \mathrm{e}^{ipx} \frac{\not{x}}{x^2 - i\varepsilon} \times \exp\left[\frac{1}{2} \ln\sqrt{-e^2x^2/4\pi} + \frac{1}{2} K_0\left(\sqrt{-e^2x^2/\pi}\right)\right] , \quad (27)$$

which entirely agrees with the formulae (1) and (2) in the case when x is spacelike. For timelike x we have to perform in (25) a rotation in the complex plane of p^2 obtaining the first formula of (2). This proves the convergence and correctness of the perturbation series in the Schwinger Model (at least for the electron propagator).

Acknowledgements. The author would like to thank to Professor J. Namysłowski for the interesting discussions.

References

1. F.J. Dyson, Phys. Rev. **85**, 631 (1952)
2. F. Calogero, N. Cim. **30**, 916 (1963)
3. J.S. Langer, Ann. Phys. **41**, 108 (1967)
4. S. Graffi, V. Grecchi, B. Simon, Phys. Lett. **32B**, 631 (1970)
5. C.M. Bender, T.T. Wu, Phys. Rev. Lett. **27**, 461 (1971)
6. D.V. Shirkov, Lett. N. Cim. **18**, 452 (1977)
7. L.N. Lipatov, Sov. Phys. JETP **72**, 411 (1977)
8. E. Brézin, J.C. Le Guillou, J. Zinn-Justin, Phys. Rev. **D15**, 1544 and 1558 (1977)
9. W.-C. Ng, W.-B. Yeung, Lett. N. Cim. **23**, 413 (1978)
10. J.C. Le Guillou, J. Zinn-Justin (eds.), *Large-Order Behaviour of Perturbation Theory* (North Holland, 1990)
11. C. Bachas, Theor. Math. Phys. **95**, 491 (1993)
12. C. Itzykson, B. Parisi, J.-B. Zuber, Phys. Rev. **D16**, 996 (1977); R. Balian et al., Phys. Rev. **D17**, 1041 (1978)
13. E.B. Bogomolny, V.A. Fateyev, Phys. Lett. **76B**, 210 (1978)
14. J. Fischer, Int. J. Mod. Phys. **A12**, 3625 (1997)
15. D.I. Kazakov, D.V. Shirkov, Fortschr. Phys. **28**, 465 (1980)
16. J. Zinn-Justin, Phys. Rep. **70**, 109 (1981)
17. I.W. Herbst, B. Simon, Phys. Lett. **78 B**, 304 (1978); **80B**, 433 (1979)
18. F. Calogero, Lett. N. Cim **25**, 533 (1979)
19. S.N. Behera, A. Khare, Phys. Lett. **2**, 169 (1981)
20. J.M. Jauch, F. Rohrlich, *The theory of photons and electrons* (Springer, New York 1971)
21. T. Radożycki, I. Białynicki-Birula, Phys. Rev. **D52**, 2439 (1995)
22. A.L. Fetter, J.D. Walecka, *Quantum theory of many-particle system* (McGraw-Hill, New York 1971)
23. K. Johnson, M. Baker, R. Willey, Phys. Rev. **136**, B111 (1964)
24. K. Johnson, R. Willey, M. Baker, Phys. Rev. **183**, 1292 (1969)
25. M. Baker, K. Johnson, Phys. Rev. **D3**, 2516 and 2541 (1971)
26. S.L. Adler, Phys. Rev. **D5**, 3021 (1972)
27. K. Johnson, M. Baker, Phys. Rev. **D8**, 1110 (1973)
28. I. Białynicki-Birula, Phys. Rev. Lett. **5**, 584 (1961)
29. I. Białynicki-Birula, Phys. Rev. **122**, 1942 (1961)
30. I. Białynicki-Birula, in *Functional Integration, Geometry and Strings*, ed. by Z. Haba, J. Sobczyk (Birkhäuser Verlag, Basel-Boston-Berlin 1989)
31. J. Schwinger, in *Theoretical Physics*, Trieste Lectures 1962 (I.A.E.A., Vienna 1963), p. 89; Phys. Rev. **128**, 2425 (1962)
32. J.-P. Eckmann, J. Magnen, R. Sénéor, Commun. Math. Phys. **39**, 251 (1975)
33. M. Fry, Phys. Lett. **B80**, 65 (1978)
34. A.R. Zhitnitsky, Phys. Rev. **D53**, 5821 (1996)
35. J. Spanier, K.B. Oldham, *An atlas of functions* (HPC-Springer, Washington, Berlin 1987)
36. I.O. Stamatescu, T.T. Wu, Nucl. Phys. **B143**, 503 (1978)
37. O. Schnetz, Ph. D. Thesis, Nurnberg 1995
38. It is worthy noting that the iteration equation (4.7) of [36] is a kind of Dyson-Schwinger equation which follows from the very construction of the diagram on Fig. 4, where $2n$-th vertex is fixed. It is also reflected by the presence of p_+ on the left hand side of (4.8). This gives in fact the expansion of the derivative of S (in coordinate space), as also seen in (4.10), whereas our equation (15) gives the expansion of S itself. Of course both approaches are equivalent.
39. I.S. Gradshteyn's, I.M. Ryzhik's, *Table of Integrals Series and Products* (Academic Press, New York 1980)
40. K. Stam, J. Phys. G: Nucl. Phys. **9**, L229 (1983). Please note the slight disagreement in coefficient